AP FRQ Review – Mr. Rich **Name:_______________________________ Per:_____ Seat:____**

AP Calculus AB

Topic 10: Functions, Miscellaneous

Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

(a) $\lim\limits_{x\to 0^-}(1 - 2\sin x) = 1$

$\lim\limits_{x\to 0^+} e^{-4x} = 1$

$f(0) = 1$

So, $\lim\limits_{x\to 0} f(x) = f(0)$.

Therefore, f is continuous at $x = 0$.

2 : analysis

(b) $f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$

$-2\cos x \ne -3$ for all values of $x < 0$.

$-4e^{-4x} = -3$ when $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) > 0$.

Therefore, $f'(x) = -3$ for $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)$.

$3 : \begin{cases} 2 : f'(x) \\ 1 : \text{value of } x \end{cases}$

(c) $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

$= \int_{-1}^{0}(1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx$

$= \Big[x + 2\cos x\Big]_{x=-1}^{x=0} + \Big[-\frac{1}{4}e^{-4x}\Big]_{x=0}^{x=1}$

$= (3 - 2\cos(-1)) + \left(-\frac{1}{4}e^{-4} + \frac{1}{4}\right)$

Average value $= \frac{1}{2}\int_{-1}^{1} f(x)\,dx$

$= \frac{13}{8} - \cos(-1) - \frac{1}{8}e^{-4}$

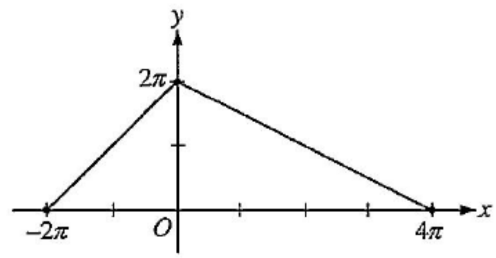
$4 : \begin{cases} 1 : \int_{-1}^{0}(1 - 2\sin x)\,dx \text{ and } \int_{0}^{1} e^{-4x}\,dx \\ 2 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

Let g be the piecewise-linear function defined on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.

(a) Find $\int_{-2\pi}^{4\pi} f(x)\,dx$. Show the computations that lead to your answer.

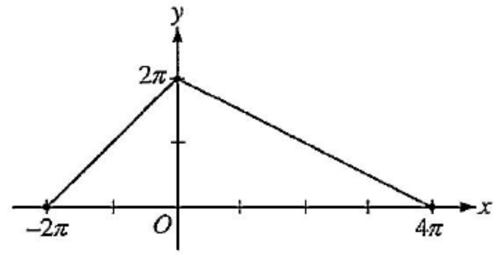
(b) Find all x-values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point.

(c) Let $h(x) = \int_0^{3x} g(t)\,dt$. Find $h'\left(-\frac{\pi}{3}\right)$.



Graph of g

Let g be the piecewise-linear function defined on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.



Graph of g

(a) Find $\int_{-2\pi}^{4\pi} f(x)\,dx$. Show the computations that lead to your answer.

(b) Find all x-values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point.

(c) Let $h(x) = \int_0^{3x} g(t)\,dt$. Find $h'\left(-\frac{\pi}{3}\right)$.

(a) $\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi} \left(g(x) - \cos\left(\frac{x}{2}\right)\right) dx$

$= 6\pi^2 - \left[2\sin\left(\frac{x}{2}\right)\right]_{x=-2\pi}^{x=4\pi}$

$= 6\pi^2$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) $f'(x) = g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right) = \begin{cases} 1 + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } -2\pi < x < 0 \\ -\frac{1}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } 0 < x < 4\pi \end{cases}$

$f'(x)$ does not exist at $x = 0$.
For $-2\pi < x < 0$, $f'(x) \neq 0$.
For $0 < x < 4\pi$, $f'(x) = 0$ when $x = \pi$.

f has critical points at $x = 0$ and $x = \pi$.

$4 : \begin{cases} 1 : \frac{d}{dx}\left(\cos\left(\frac{x}{2}\right)\right) \\ 1 : g'(x) \\ 1 : x = 0 \\ 1 : x = \pi \end{cases}$

(c) $h'(x) = g(3x) \cdot 3$

$h'\left(-\frac{\pi}{3}\right) = 3g(-\pi) = 3\pi$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

The function f is defined by $f(x)=\sqrt{25-x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x=-3$.

(c) Let g be the function defined by $g(x)=\begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x=-3$? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25-x^2}\,dx$.

(a) $f'(x)=\frac{1}{2}\left(25-x^2\right)^{-1/2}(-2x)=\frac{-x}{\sqrt{25-x^2}}, \quad -5 < x < 5$

$2: f'(x)$

(b) $f'(-3)=\frac{3}{\sqrt{25-9}}=\frac{3}{4}$

$f(-3)=\sqrt{25-9}=4$

An equation for the tangent line is $y=4+\frac{3}{4}(x+3)$.

$2: \begin{cases} 1: f'(-3) \\ 1: \text{answer} \end{cases}$

(c) $\lim\limits_{x\to -3^-} g(x)=\lim\limits_{x\to -3^-} f(x)=\lim\limits_{x\to -3^-}\sqrt{25-x^2}=4$

$\lim\limits_{x\to -3^+} g(x)=\lim\limits_{x\to -3^+}(x+7)=4$

Therefore, $\lim\limits_{x\to -3} g(x)=4$.

$g(-3)=f(-3)=4$

So, $\lim\limits_{x\to -3} g(x)=g(-3)$.

Therefore, g is continuous at $x=-3$.

$2: \begin{cases} 1: \text{considers one-sided limits} \\ 1: \text{answer with explanation} \end{cases}$

(d) Let $u=25-x^2 \Rightarrow du=-2x\,dx$

$\int_0^5 x\sqrt{25-x^2}\,dx=-\frac{1}{2}\int_{25}^{0}\sqrt{u}\,du$

$=\left[-\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right]_{u=25}^{u=0}$

$=-\frac{1}{3}(0-125)=\frac{125}{3}$

$3: \begin{cases} 2: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

(a) $\dfrac{A(30) - A(0)}{30 - 0} = -0.197$ (or -0.196) lbs/day	1 : answer with units
(b) $A'(15) = -0.164$ (or -0.163) The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t = 15$ days.	$2 : \begin{cases} 1 : A'(15) \\ 1 : \text{interpretation} \end{cases}$
(c) $A(t) = \frac{1}{30}\int_0^{30} A(t)\,dt \Rightarrow t = 12.415$ (or 12.414)	$2 : \begin{cases} 1 : \frac{1}{30}\int_0^{30} A(t)\,dt \\ 1 : \text{answer} \end{cases}$
(d) $L(t) = A(30) + A'(30)\cdot(t - 30)$ $A'(30) = -0.055976$ $A(30) = 0.782928$ $L(t) = 0.5 \Rightarrow t = 35.054$	$4 : \begin{cases} 2 : \text{expression for } L(t) \\ 1 : L(t) = 0.5 \\ 1 : \text{answer} \end{cases}$

Let f be the function given by $f(x) = 2xe^{2x}$.

(a) Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

(c) What is the range of f?

(d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

Let f be the function given by $f(x) = 2xe^{2x}$.

(a) Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

(c) What is the range of f?

(d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

(a) $\lim_{x\to-\infty} 2xe^{2x} = 0$

$\lim_{x\to\infty} 2xe^{2x} = \infty$ or DNE

$2 \begin{cases} 1: & 0 \text{ as } x \to -\infty \\ 1: & \infty \text{ or DNE as } x \to \infty \end{cases}$

(b) $f'(x) = 2e^{2x} + 2x \cdot 2 \cdot e^{2x} = 2e^{2x}(1+2x) = 0$

if $x = -1/2$

$f(-1/2) = -1/e$ or -0.368 or -0.367

$-1/e$ is an absolute minimum value because:

(i) $f'(x) < 0$ for all $x < -1/2$ and
$f'(x) > 0$ for all $x > -1/2$

–or–

(ii) $f'(x)$: $-$ for $x < -1/2$, $+$ for $x > -1/2$ (sign chart at $-1/2$)

and $x = -1/2$ is the only critical number

3:
- 1: solves $f'(x) = 0$
- 1: evaluates f at student's critical point
  0/1 if not local minimum from student's derivative
- 1: justifies absolute minimum value
  0/1 for a local argument
  0/1 without explicit symbolic derivative

Note: 0/3 if no absolute minimum based on student's derivative

(c) Range of $f = [-1/e, \infty)$
or $[-0.367, \infty)$
or $[-0.368, \infty)$

1: answer

Note: must include the left–hand endpoint; exclude the right–hand "endpoint"

(d) $y' = be^{bx} + b^2xe^{bx} = be^{bx}(1+bx) = 0$

if $x = -1/b$

At $x = -1/b$, $y = -1/e$

y has an absolute minimum value of $-1/e$ for all nonzero b

3:
- 1: sets $y' = be^{bx}(1+bx) = 0$
- 1: solves student's $y' = 0$
- 1: evaluates y at a critical number and gets a value independent of b

Note: 0/3 if only considering specific values of b

A cubic polynomial function f is defined by

$$f(x) = 4x^3 + ax^2 + bx + k$$

where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$.

(a) Find the values of a and b.

(b) If $\int_0^1 f(x)\,dx = 32$, what is the value of k?

A cubic polynomial function f is defined by

$$f(x) = 4x^3 + ax^2 + bx + k$$

where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$.

(a) Find the values of a and b.

(b) If $\int_0^1 f(x)\,dx = 32$, what is the value of k?

(a) $f'(x) = 12x^2 + 2ax + b$

$f''(x) = 24x + 2a$

$f'(-1) = 12 - 2a + b = 0$

$f''(-2) = -48 + 2a = 0$

$a = 24$

$b = -12 + 2a = 36$

$$5 : \begin{cases} 1 : f'(x) \\ 1 : f''(x) \\ 1 : f'(-1) = 0 \\ 1 : f''(-2) = 0 \\ 1 : a,\ b \end{cases}$$

(b) $\int_0^1 \left(4x^3 + 24x^2 + 36x + k\right)dx$

$= x^4 + 8x^3 + 18x^2 + kx\Big|_{x=0}^{x=1} = 27 + k$

$27 + k = 32$

$k = 5$

$$4 : \begin{cases} 2 : \text{antidifferentiation} \\ \quad < -1 > \text{ each error} \\ 1 : \text{expression in } k \\ 1 : k \end{cases}$$

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$

$h'(x)$	$-$	0	$+$	und	$-$	0	$+$
x		$-\sqrt{2}$		0		$\sqrt{2}$	

Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

4 : $\begin{cases} 1 : x = \pm\sqrt{2} \\ 1 : \text{analysis} \\ 2 : \text{conclusions} \\ \quad <-1> \text{ not dealing with discontinuity at } 0 \end{cases}$

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

3 : $\begin{cases} 1 : h''(x) \\ 1 : h''(x) > 0 \\ 1 : \text{answer} \end{cases}$

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

1 : answer with reason

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

(a) f is continuous at $x = 3$ because

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2.$

Therefore, $\lim_{x \to 3} f(x) = 2 = f(3)$.

$2: \begin{cases} 1: \text{answers "yes" and equates the values of the left- and right-hand limits} \\ 1: \text{explanation involving limits} \end{cases}$

(b)
$$\begin{aligned} \int_0^5 f(x)\,dx &= \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx \\ &= \frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{1}{2}x^2\right)\Big|_3^5 \\ &= \left(\frac{16}{3} - \frac{2}{3}\right) + \left(\frac{25}{2} - \frac{21}{2}\right) = \frac{20}{3} \end{aligned}$$

Average value: $\frac{1}{5}\int_0^5 f(x)\,dx = \frac{4}{3}$

$4: \begin{cases} 1: k\int_0^3 f(x)\,dx + k\int_3^5 f(x)\,dx \\ \quad (\text{where } k \ne 0) \\ 1: \text{antiderivative of } \sqrt{x+1} \\ 1: \text{antiderivative of } 5 - x \\ 1: \text{evaluation and answer} \end{cases}$

(c) Since g is continuous at $x = 3$, $2k = 3m + 2$.

$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}} & \text{for } 0 < x < 3 \\ m & \text{for } 3 < x < 5 \end{cases}$$

$\lim_{x \to 3^-} g'(x) = \frac{k}{4}$ and $\lim_{x \to 3^+} g'(x) = m$

Since these two limits exist and g is differentiable at $x = 3$, the two limits are equal. Thus $\frac{k}{4} = m$.

$8m = 3m + 2;\ m = \frac{2}{5}$ and $k = \frac{8}{5}$

$3: \begin{cases} 1: 2k = 3m + 2 \\ 1: \frac{k}{4} = m \\ 1: \text{values for } k \text{ and } m \end{cases}$

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

(a) $h'(2) = 2/3$ From tangent line equation

(b) $a(x) = 3x^3h(x)$

$a'(x) = 3x^3h'(x) + 9x^2h(x)$

$a'(2) = 3(2)^3h'(2) + 9(2)^2h(2)$

$= 3(8)(2/3) + 9(4)(4) = 16 + 144 = \underline{160}$

(c) $h(x) = \dfrac{x^2-4}{1-(f(x))^3}$ for $x \neq 2$

If $\lim_{x\to 2} h(x)$ can be solved using L'Hopital's Rule, then

$1 - [f(2)]^3 = 0 \Rightarrow \underline{f(2) = 1}$

But also, since $h(x)$ is twice-differentiable it must also be continuous, so:

$\lim_{x\to 2} h(x) = h(2) = 4$

So, by L'Hopital's Rule, $\lim_{x\to 2} \dfrac{x^2-4}{1-[f(x)]^3} = \lim_{x\to 2} \dfrac{2x}{-3f(x)^2f'(x)}$

and $\lim_{x\to 2} \dfrac{2x}{-3f(x)^2f'(x)} = \dfrac{2(2)}{-3f(2)^2f'(2)} = 4$

$\dfrac{4}{(-3)(1)^2f'(2)} = 4 \Rightarrow f'(2) = \dfrac{4}{(-3)(4)} = -\dfrac{1}{3}$

So $f(2) = 1$, $f'(2) = -\frac{1}{3}$