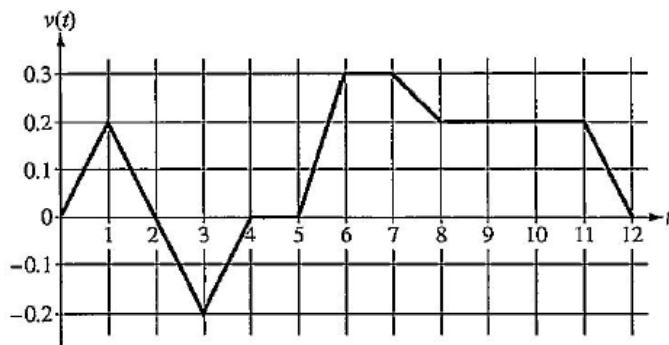


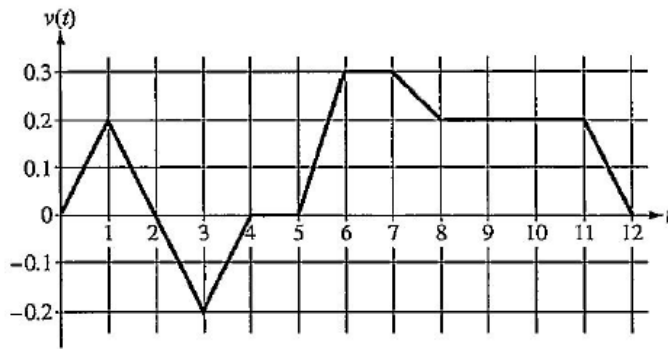
AP Calculus AB

Topic 6: Accumulation Functions



Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \leq t \leq 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

- (a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.
- (b) Using correct units, explain the meaning of $\int_0^{12} |v(t)| dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)| dt$.
- (c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.
- (d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15} \sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \leq t \leq 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.



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- (b) Using correct units, explain the meaning of $\int_0^{12} |v(t)| dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)| dt$.
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(a) $a(7.5) = v'(7.5) = \frac{v(8) - v(7)}{8 - 7} = -0.1$ miles/minute²

(b) $\int_0^{12} |v(t)| dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$.

$$\int_0^{12} |v(t)| dt = \int_0^2 v(t) dt - \int_2^4 v(t) dt + \int_4^{12} v(t) dt$$

$$= 0.2 + 0.2 + 1.4 = 1.8 \text{ miles}$$

(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.

(d) $\int_0^{12} w(t) dt = 1.6$; Larry lives 1.6 miles from school.

$\int_0^{12} v(t) dt = 1.4$; Caren lives 1.4 miles from school.

Therefore, Caren lives closer to school.

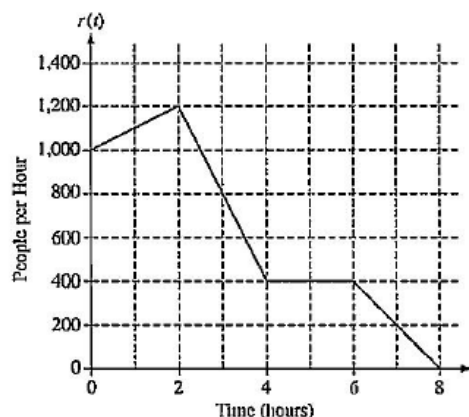
2 : { 1 : answer
1 : units

2 : { 1 : meaning of integral
1 : value of integral

2 : { 1 : answer
1 : reason

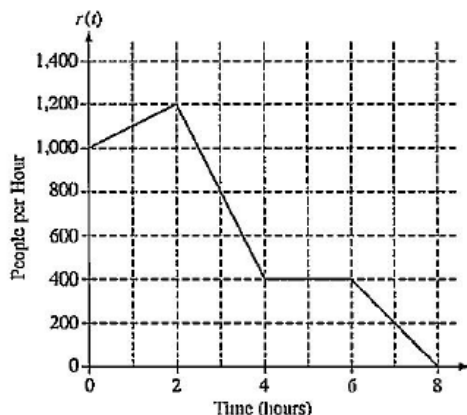
3 : { 2 : Larry's distance from school
1 : integral
1 : value
1 : Caren's distance from school and conclusion

There are 700 people in line for a popular amusement-park ride when the ride begins operation in the morning. Once it begins operation, the ride accepts passengers until the park closes 8 hours later. While there is a line, people move onto the ride at a rate of 800 people per hour. The graph above shows the rate, $r(t)$, at which people arrive at the ride throughout the day. Time t is measured in hours from the time the ride begins operation.



- How many people arrive at the ride between $t = 0$ and $t = 3$? Show the computations that lead to your answer.
- Is the number of people waiting in line to get on the ride increasing or decreasing between $t = 2$ and $t = 3$? Justify your answer.
- At what time t is the line for the ride the longest? How many people are in line at that time? Justify your answers.
- Write, but do not solve, an equation involving an integral expression of r whose solution gives the earliest time t at which there is no longer a line for the ride.

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- (a) How many people arrive at the ride between $t = 0$ and $t = 3$? Show the computations that lead to your answer.
- (b) Is the number of people waiting in line to get on the ride increasing or decreasing between $t = 2$ and $t = 3$? Justify your answer.
- (c) At what time t is the line for the ride the longest? How many people are in line at that time? Justify your answers.
- (d) Write, but do not solve, an equation involving an integral expression of r whose solution gives the earliest time t at which there is no longer a line for the ride.

(a) $\int_0^3 r(t) dt = 2 \cdot \frac{1000 + 1200}{2} + \frac{1200 + 800}{2} = 3200$ people

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

- (b) The number of people waiting in line is increasing because people move onto the ride at a rate of 800 people per hour and for $2 < t < 3$, $r(t) > 800$.

1 : answer with reason

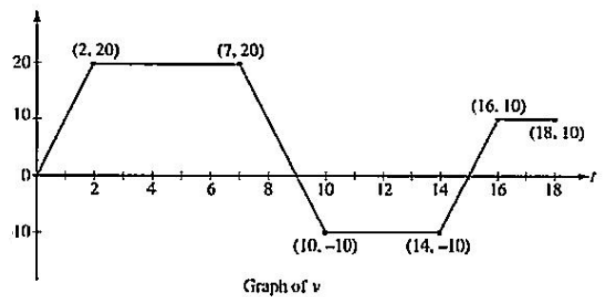
- (c) $r(t) = 800$ only at $t = 3$
 For $0 \leq t < 3$, $r(t) > 800$. For $3 < t \leq 8$, $r(t) < 800$.
 Therefore, the line is longest at time $t = 3$.
 There are $700 + 3200 - 800 \cdot 3 = 1500$ people waiting in line at time $t = 3$.

3 : $\begin{cases} 1 : \text{identifies } t = 3 \\ 1 : \text{number of people in line} \\ 1 : \text{justification} \end{cases}$

(d) $0 = 700 + \int_0^t r(s) ds - 800t$

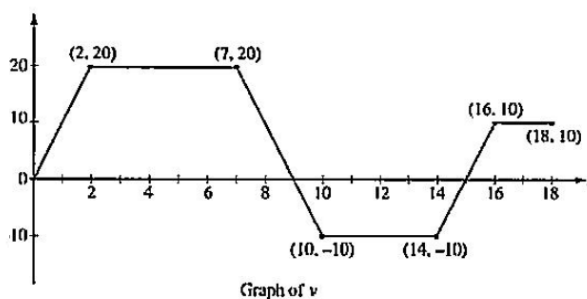
3 : $\begin{cases} 1 : 800t \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

A squirrel starts at building A at time $t = 0$ and travels along a straight wire connected to building B . For $0 \leq t \leq 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.



- (a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.
- (b) At what time in the interval $0 \leq t \leq 18$ is the squirrel farthest from building A ? How far from building A is the squirrel at this time?
- (c) Find the total distance the squirrel travels during the time interval $0 \leq t \leq 18$.
- (d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

A squirrel starts at building A at time $t = 0$ and travels along a straight wire connected to building B . For $0 \leq t \leq 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.



- (a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.
- (b) At what time in the interval $0 \leq t \leq 18$ is the squirrel farthest from building A ? How far from building A is the squirrel at this time?
- (c) Find the total distance the squirrel travels during the time interval $0 \leq t \leq 18$.
- (d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

(a) The squirrel changes direction whenever its velocity changes sign. This occurs at $t = 9$ and $t = 15$.

(b) Velocity is 0 at $t = 0$, $t = 9$, and $t = 15$.

t	position at time t
0	0
9	$\frac{9+5}{2} \cdot 20 = 140$
15	$140 - \frac{6+4}{2} \cdot 10 = 90$
18	$90 + \frac{3+2}{2} \cdot 10 = 115$

The squirrel is farthest from building A at time $t = 9$; its greatest distance from the building is 140.

(c) The total distance traveled is $\int_0^{18} |v(t)| dt = 140 + 50 + 25 = 215$.

2 : $\begin{cases} 1 : t\text{-values} \\ 1 : \text{explanation} \end{cases}$

2 : $\begin{cases} 1 : \text{identifies candidates} \\ 1 : \text{answers} \end{cases}$

1 : answer

(d) For $7 < t < 10$, $a(t) = \frac{20 - (-10)}{7 - 10} = -10$

$$v(t) = 20 - 10(t - 7) = -10t + 90$$

$$x(7) = \frac{7+5}{2} \cdot 20 = 120$$

$$x(t) = x(7) + \int_7^t (-10u + 90) du$$

$$= 120 + (-5u^2 + 90u) \Big|_{u=7}^{u=t}$$

$$= -5t^2 + 90t - 265$$

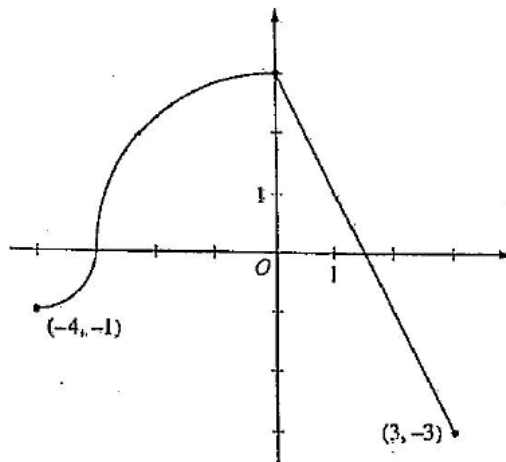
4 : $\begin{cases} 1 : a(t) \\ 1 : v(t) \\ 2 : x(t) \end{cases}$

The continuous function f is defined on the interval $-4 \leq x \leq 3$.

The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

$$\text{Let } g(x) = 2x + \int_0^x f(t) dt.$$

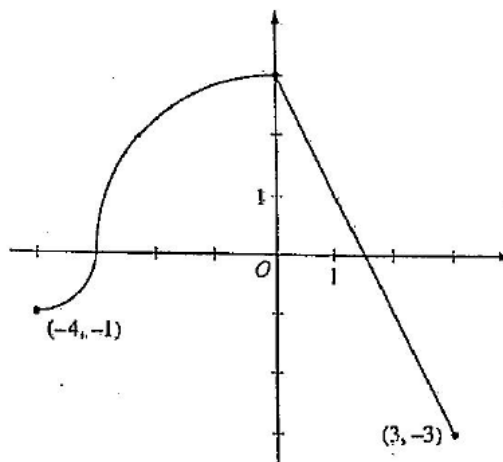
- (a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.
- (b) Determine the x -coordinate of the point at which g has an absolute maximum on the interval $-4 \leq x \leq 3$. Justify your answer.
- (c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.
- (d) Find the average rate of change of f on the interval $-4 \leq x \leq 3$. There is no point c , $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.



Graph of f

The continuous function f is defined on the interval $-4 \leq x \leq 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

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Graph of f

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- (b) Determine the x -coordinate of the point at which g has an absolute maximum on the interval $-4 \leq x \leq 3$. Justify your answer.
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- (d) Find the average rate of change of f on the interval $-4 \leq x \leq 3$. There is no point c , $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t) dt = -6 - \frac{9\pi}{4}$
 $g'(x) = 2 + f(x)$
 $g'(-3) = 2 + f(-3) = 2$

3 : $\begin{cases} 1 : g(-3) \\ 1 : g'(x) \\ 1 : g'(-3) \end{cases}$

(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$.
 $g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$.
 Therefore, g has an absolute maximum at $x = \frac{5}{2}$.

3 : $\begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : \text{identifies interior candidate} \\ 1 : \text{answer with justification} \end{cases}$

(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus, the graph of g has a point of inflection at $x = 0$.

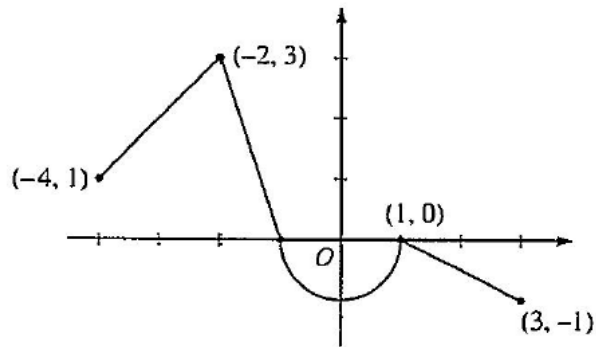
1 : answer with reason

(d) The average rate of change of f on the interval $-4 \leq x \leq 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = \frac{2}{7}$.

2 : $\begin{cases} 1 : \text{average rate of change} \\ 1 : \text{explanation} \end{cases}$

To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.

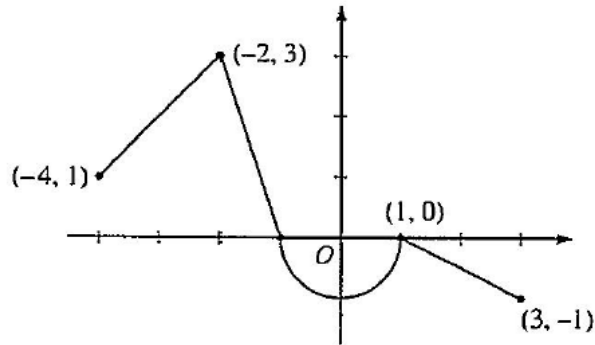
Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t) dt$.



Graph of f

- Find the values of $g(2)$ and $g(-2)$.
- For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.
- Find the x -coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.
- For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

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Graph of f

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- (b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.
- (c) Find the x -coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.
- (d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t) dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$
 $g(-2) = \int_1^{-2} f(t) dt = -\int_{-2}^1 f(t) dt$
 $= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

2 : $\begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$

(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$
 $g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$

2 : $\begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$

(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$.
 Therefore, g has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

3 : $\begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

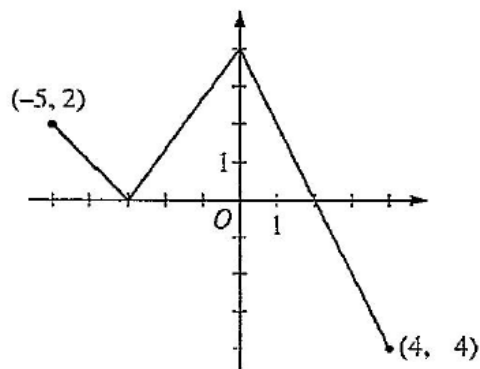
(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

Let g be the function defined by $g(x) = \int_{-3}^x f(t) dt$.

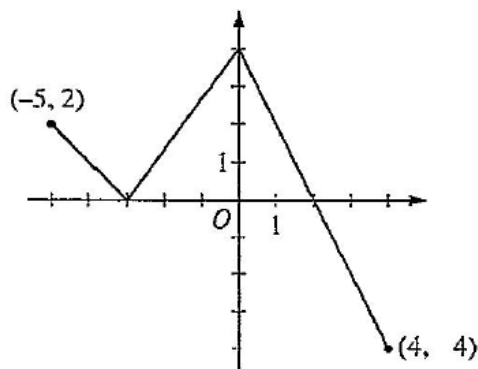
- Find $g(3)$.
- On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.
- The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.
- The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.



Graph of f

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

Let g be the function defined by $g(x) = \int_{-3}^x f(t) dt$.



Graph of f

- (a) Find $g(3)$.
- (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.
- (c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.
- (d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

(a) $g(3) = \int_{-3}^3 f(t) dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \frac{5xg'(x) - g(x)5}{(5x)^2} = \frac{5xg'(x) - 5g(x)}{25x^2}$

3 : $\begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

$$h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2}$$

$$= \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3}$$

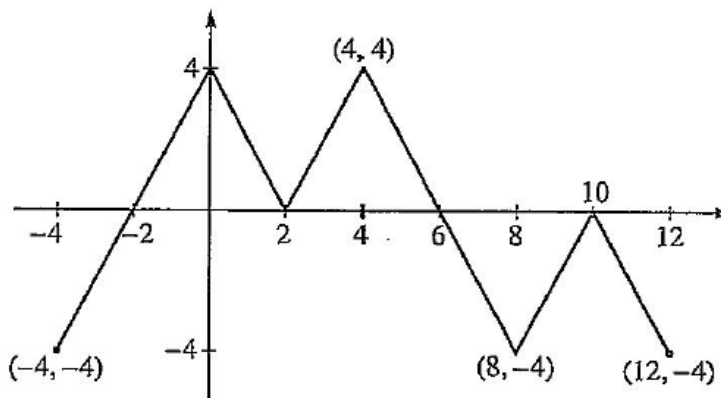
(d) $p'(x) = f'(x^2 - x)(2x - 1)$

3 : $\begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

$$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$$

The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t) dt$.

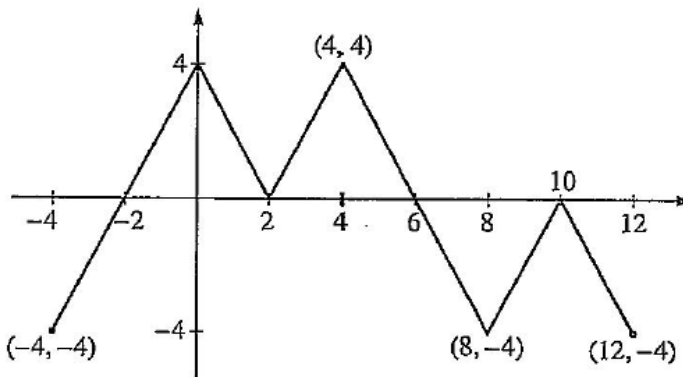
- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.



Graph of f

The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$,

the function g is defined by $g(x) = \int_2^x f(t) dt$.



Graph of f

- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

- (a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \leq 0$ for $8 \leq x \leq 12$.
- (b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \leq x \leq 4$ and decreasing for $4 \leq x \leq 8$.
- (c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \leq x \leq 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

- (d) $g(x) \leq 0$ for $-4 \leq x \leq 2$ and $10 \leq x \leq 12$.

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

1 : answer with justification

1 : answer with justification

4 : { 1 : considers $x = -2$ and $x = 6$ as candidates
1 : considers $x = -4$ and $x = 12$
2 : answers with justification

2 : intervals

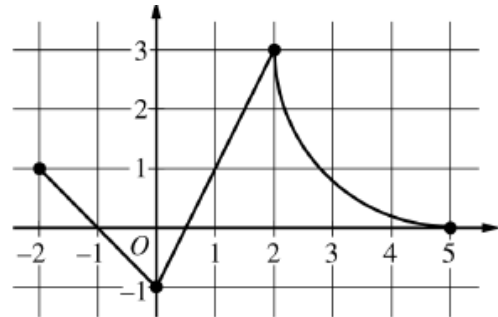
The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f , consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f .

(a) If $\int_{-6}^5 f(x) dx = 7$, find the value of $\int_{-6}^{-2} f(x) dx$. Show the work that leads to your answer.

(b) Evaluate $\int_3^5 (2f'(x) + 4) dx$.

(c) The function g is given by $g(x) = \int_{-2}^x f(t) dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

(d) Find $\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.



Graph of f

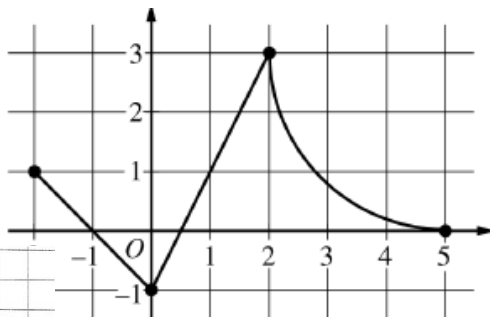
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(a) If $\int_{-6}^5 f(x) dx = 7$, find the value of $\int_{-2}^5 f(x) dx$. Show the work that leads to your answer.

(b) Evaluate $\int_3^5 (2f'(x) + 4) dx$.

(c) The function g is given by $g(x) = \int_{-2}^x f(t) dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

(d) Find $\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.



Graph of f

(a)
$$\int_{-6}^{-2} f(x) dx = \int_{-6}^5 f(x) dx + \int_5^{-2} f(x) dx$$

$$= \int_{-6}^5 f(x) dx - \int_{-2}^5 f(x) dx$$

Note: On interval $[-2, 1]$, the area nets to zero.

$$= 7 - \left[\underbrace{\frac{1+3}{2}(1)}_{\text{Trapezoid } [1,2]} + \underbrace{(3)(3)}_{\text{Square } [2,5]} - \underbrace{\frac{\pi(3)^2}{4}}_{\text{Quarter Circle } [2,5]} \right] = -4 + \frac{9\pi}{4}$$

(b)
$$\int_3^5 (2f'(x) + 4) dx = 2 \int_3^5 f'(x) dx + \int_3^5 4 dx$$

$$= 2[f(5) - f(3)] + 4x \Big|_3^5$$

$$= 2[0 - (3 - \sqrt{5})] + 8 = 2(-3 + \sqrt{5}) + 8$$

$$= -6 + 2\sqrt{5} + 8 = 2 + 2\sqrt{5}$$

(c) $g(x) = \int_{-2}^x f(t) dt$ (crit. points where $g'(x) = 0$)

$g'(x) = f(x)$ $f(x) = 0$ at $x = -1, 1/2, 5$

x	g(x)
-2	0
-1	0.5
1/2	-0.25
5	$7 + 4 - 9\pi/4 = 11 - 9\pi/4$

Absolute max is at critical point or endpoint.

Absolute max of $g(x)$ from $[-2, 5]$ is $11 - 9\pi/4$.

(d) $\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10 - 3(2)}{1 - \pi/4} = \frac{4}{1 - \pi/4}$