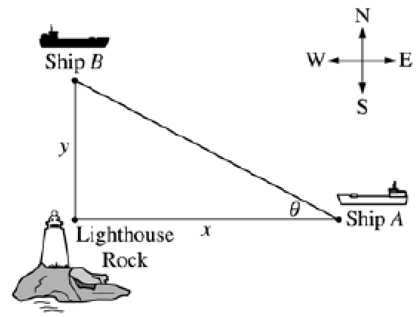
AP FRQ Review – Mr. Rich **Name:_______________________________ Per:_____ Seat:____**

AP Calculus AB

Topic 5: Related Rates (Some Multiple Choice Included Also)



Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

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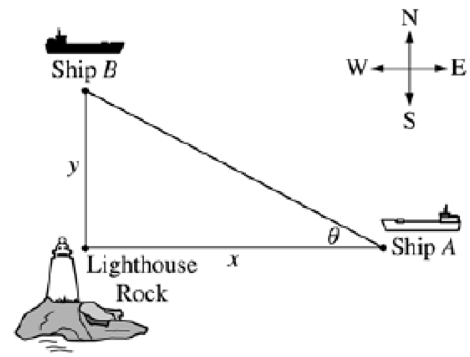

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(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

(a) Distance $= \sqrt{3^2 + 4^2} = 5$ km

1 : answer

(b) $r^2 = x^2 + y^2$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or explicitly:

$$r = \sqrt{x^2 + y^2}$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

At $x = 4$, $y = 3$,

$$\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6 \text{ km/hr}$$

4 $\begin{cases} 1 : \text{expression for distance} \\ 2 : \text{differentiation with respect to } t \\ \quad < -2 > \text{ chain rule error} \\ 1 : \text{evaluation} \end{cases}$

(c) $\tan\theta = \dfrac{y}{x}$

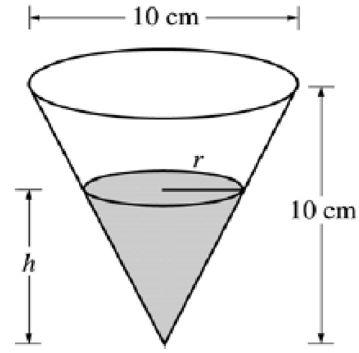
$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

At $x = 4$ and $y = 3$, $\sec\theta = \dfrac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4) - (-15)(3)}{16}\right)$$

$$= \frac{85}{25} = \frac{17}{5} \text{ radians/hr}$$

4 $\begin{cases} 1 : \text{expression for } \theta \text{ in terms of } x \text{ and } y \\ 2 : \text{differentiation with respect to } t \\ \quad < -2 > \text{ chain rule, quotient rule, or} \\ \qquad \text{transcendental function error} \\ \quad \text{note: } 0/2 \text{ if no trig or inverse trig} \\ \qquad \text{function} \\ 1 : \text{evaluation} \end{cases}$



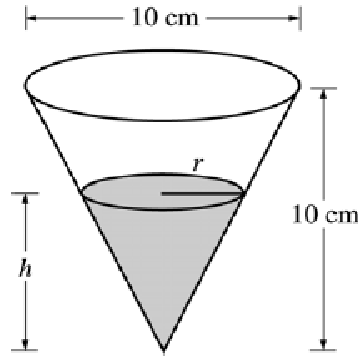
A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?



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(a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi \text{ cm}^3$

(b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3;\ \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi \ \text{cm}^3/\text{hr}$$

OR

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right);\ \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5,r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left(\frac{25}{4}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$$

$$= -\frac{15}{8}\pi \text{ cm}^3/\text{hr}$$

(c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} = -\frac{3}{40}\pi h^2$

$$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10}\cdot \text{area}$$

The constant of proportionality is $-\frac{3}{10}$.

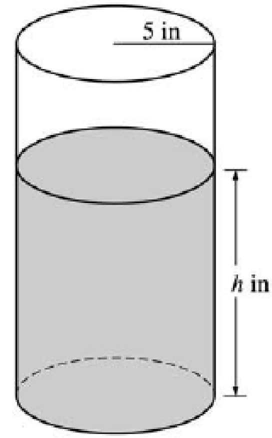
Scoring:

1 : V when $h = 5$

5 {
- 1 : $r = \frac{1}{2}h$ in (a) or (b)
- 1 : { V as a function of one variable in (a) or (b) OR $\frac{dr}{dt}$ }
- 2 : $\frac{dV}{dt}$
  < −2 > chain rule or product rule error
- 1 : evaluation at $h = 5$

2 {
- 1 : shows $\frac{dV}{dt} = k \cdot \text{area}$
- 1 : identifies constant of proportionality

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

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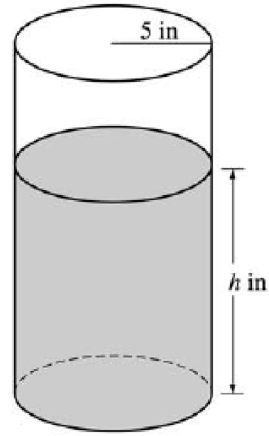

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(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

$$3 : \begin{cases} 1 : \dfrac{dV}{dt} = -5\pi\sqrt{h} \\ 1 : \text{computes } \dfrac{dV}{dt} \\ 1 : \text{shows result} \end{cases}$$

(b) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}dh = -\frac{1}{5}dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

$$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } h = 17 \\ \quad\ \text{when } t = 0 \\ 1 : \text{solves for } h \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\dfrac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

At time t, $t \geq 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. (The volume V of a sphere with a radius r is $V = \frac{4}{3}\pi r^3$.)

(a) Find the radius of the sphere as a function of t.

(b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$?

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(a) $\frac{dV}{dt} = \frac{k}{r}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{k}{r} = 4\pi r^2 \frac{dr}{dt}$

$k\,dt = 4\pi r^3\,dr$

$kt + C = \pi r^4$

At $t=0$, $r=1$, so $C = \pi$

At $t=15$, $r=2$, so $15k + \pi = 16\pi$, $k = \pi$

$\pi r^4 = \pi t + \pi$

$r = \sqrt[4]{t+1}$

(b) At $t=0$, $r=1$, so $V(0) = \frac{4}{3}\pi$

$27V(0) = 27\left(\frac{4}{3}\pi\right) = 36\pi$

$36\pi = \frac{4}{3}\pi r^3$

$r = 3$

$\sqrt[4]{t+1} = 3$

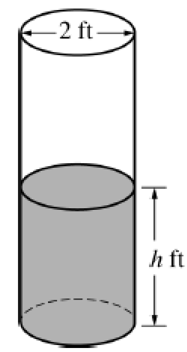
$t = 80$

A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.



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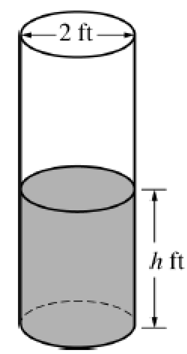


(a) $V = \pi r^2 h$ $\quad r = 1$ ft (constant)

$dV/dt = \pi r^2 \, dh/dt \qquad \left.dh/dt\right|_{h=4\text{ft}} = -\frac{1}{10}\sqrt{4} = -\frac{1}{5}$ ft/sec

$\left.dV/dt\right|_{h=4\text{ft}} = \pi(1\text{ ft})^2(-1/5 \text{ ft/sec}) = -\frac{\pi}{5} \text{ ft}^3/\text{sec}$

(b) $\frac{dh}{dt} = -\frac{1}{10}h^{1/2}$

$\frac{d^2h}{dt^2} = -\frac{1}{20}h^{-1/2} \cdot \frac{dh}{dt} = -\frac{1}{20}h^{-1/2}\left(-\frac{1}{10}h^{1/2}\right) = \frac{1}{200}$

Because d^2h/dt^2 is always positive, the rate of change of height is increasing when height is 3 ft.

(c) $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h} \Rightarrow \frac{dh}{\sqrt{h}} = -\frac{1}{10}dt$

$\int h^{-1/2} dh = \int -\frac{1}{10} dt \Rightarrow 2h^{1/2} = -\frac{1}{10}t + C$

Using $h(0) = 5$ ft: $\quad 2\sqrt{5} = C$

So $\quad 2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5} \Rightarrow \sqrt{h} = -\frac{1}{20}t + \sqrt{5}$

$h = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$

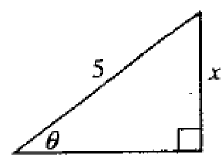
Related Rate MC Problems

26. The radius r of a sphere is increasing at the uniform rate of 0.3 inches per second. At the instant when the surface area S becomes 100π square inches, what is the rate of increase, in cubic inches per second, in the volume V? $\left(S = 4\pi r^2 \text{ and } V = \frac{4}{3}\pi r^3\right)$

(A) 10π (B) 12π (C) 22.5π (D) 25π (E) 30π

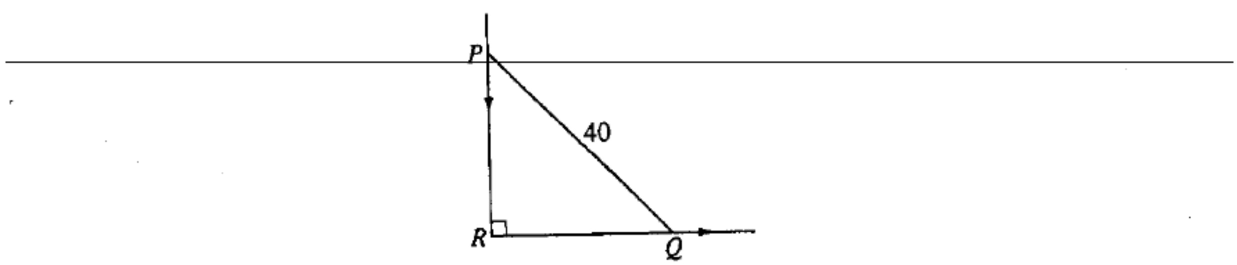
39. The radius of a circle is increasing at a nonzero rate, and at a certain instant, the rate of increase in the area of the circle is numerically equal to the rate of increase in its circumference. At this instant, the radius of the circle is

(A) $\frac{1}{\pi}$ (B) $\frac{1}{2}$ (C) $\frac{2}{\pi}$ (D) 1 (E) 2



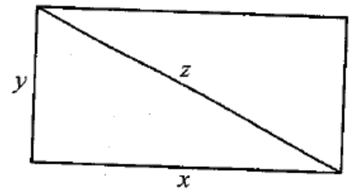
23. In the triangle shown above, if θ increases at a constant rate of 3 radians per minute, at what rate is x increasing in units per minute when x equals 3 units?

(A) 3 (B) $\frac{15}{4}$ (C) 4 (D) 9 (E) 12



34. In the figure above, PQ represents a 40-foot ladder with end P against a vertical wall and end Q on level ground. If the ladder is slipping down the wall, what is the distance RQ at the instant when Q is moving along the ground $\frac{3}{4}$ as fast as P is moving down the wall?

(A) $\frac{6}{5}\sqrt{10}$ (B) $\frac{8}{5}\sqrt{10}$ (C) $\frac{80}{\sqrt{7}}$ (D) 24 (E) 32



40. The sides of the rectangle above increase in such a way that $\frac{dz}{dt} = 1$ and $\frac{dx}{dt} = 3\frac{dy}{dt}$. At the instant when $x = 4$ and $y = 3$, what is the value of $\frac{dx}{dt}$?

(A) $\frac{1}{3}$ (B) 1 (C) 2 (D) $\sqrt{5}$ (E) 5

37. A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person is walking at a constant rate and the person's shadow is lengthening at the rate of $\frac{4}{9}$ meter per second, at what rate, in meters per second, is the person walking?

(A) $\frac{4}{27}$ (B) $\frac{4}{9}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$ (E) $\frac{16}{9}$

78. The radius of a circle is decreasing at a constant rate of 0.1 centimeter per second. In terms of the circumference C, what is the rate of change of the area of the circle, in square centimeters per second?

(A) $-(0.2)\pi C$

(B) $-(0.1)C$

(C) $-\frac{(0.1)C}{2\pi}$

(D) $(0.1)^2 C$

(E) $(0.1)^2 \pi C$

86. Let $f(x) = \sqrt{x}$. If the rate of change of f at $x = c$ is twice its rate of change at $x = 1$, then $c =$

(A) $\frac{1}{4}$ (B) 1 (C) 4 (D) $\frac{1}{\sqrt{2}}$ (E) $\frac{1}{2\sqrt{2}}$

8 90. If the base b of a triangle is increasing at a rate of 3 inches per minute while its height h is decreasing at a rate of 3 inches per minute, which of the following must be true about the area A of the triangle?

(A) A is always increasing.
(B) A is always decreasing.
(C) A is decreasing only when $b < h$.
(D) A is decreasing only when $b > h$.
(E) A remains constant.

81. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the crossing and watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

(A) 57.60 (B) 57.88 (C) 59.20 (D) 60.00 (E) 67.40

5 31. The volume of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. If the radius and the height both increase at a constant rate of $\frac{1}{2}$ centimeter per second, at what rate, in cubic centimeters per second, is the volume increasing when the height is 9 centimeters and the radius is 6 centimeters?

(A) $\frac{1}{2}\pi$ (B) 10π (C) 24π (D) 54π (E) 108π

22. The area of a circular region is increasing at a rate of 96π square meters per second. When the area of the region is 64π square meters, how fast, in meters per second, is the radius of the region increasing?

(A) 6 (B) 8 (C) 16 (D) $4\sqrt{3}$ (E) $12\sqrt{3}$

9. When the area in square units of an expanding circle is increasing twice as fast as its radius in linear units, the radius is

(A) $\frac{1}{4\pi}$ (B) $\frac{1}{4}$ (C) $\frac{1}{\pi}$ (D) 1 (E) π

34. The top of a 25-foot ladder is sliding down a vertical wall at a constant rate of 3 feet per minute. When the top of the ladder is 7 feet from the ground, what is the rate of change of the distance between the bottom of the ladder and the wall?

(A) $-\frac{7}{8}$ feet per minute

(B) $-\frac{7}{24}$ feet per minute

(C) $\frac{7}{24}$ feet per minute

(D) $\frac{7}{8}$ feet per minute

(E) $\frac{21}{25}$ feet per minute