AP FRQ Review – Mr. Rich Name:_______________________________ Per:_____ Seat:____

AP Stats Topic 5: Proportions Inferencing

Some boxes of a certain brand of breakfast cereal include a voucher for a free video rental inside the box. The company that makes the cereal claims that a voucher can be found in 20 percent of the boxes. However, based on their experiences eating this cereal at home, a group of students believes that the proportion of boxes with vouchers is less than 0.2. This group of students purchased 65 boxes of the cereal to investigate the company's claim. The students found a total of 11 vouchers for free video rentals in the 65 boxes.

Suppose it is reasonable to assume that the 65 boxes purchased by the students are a random sample of all boxes of this cereal. Based on this sample, is there support for the students' belief that the proportion of boxes with vouchers is less than 0.2 ? Provide statistical evidence to support your answer.

Solution

This question is divided into four parts.

Part (a): State a correct pair of hypotheses.

Let p = the proportion of boxes of this brand of breakfast cereal that include a voucher for a free video rental.

$$H_0 : p = 0.2$$
$$H_a : p < 0.2$$

Part (b): Identify a correct test (by name or by formula) and check appropriate conditions.

One-sample z-test for a proportion *OR* $z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1 - p_0)}{n}}}$

Conditions:

1. $np_0 = 65 \times 0.2 = 13 > 10$ and $n(1 - p_0) = 65 \times 0.8 = 52 > 10$.
2. It is reasonable to assume that the company produces more than $65 \times 10 = 650$ boxes of this cereal ($N > 10n$).
3. The observations are independent because it is reasonable to assume that the 65 boxes are a random sample of all boxes of this cereal.

Part (c): Use correct mechanics and calculations, and provide the p-value (or rejection region).

The sample proportion is $\hat{p} = \dfrac{11}{65} = 0.169$. The test statistic is $z = \dfrac{0.169 - 0.2}{\sqrt{\dfrac{0.2(1 - 0.2)}{65}}} = -0.62$ and the p-value is

$P(Z < -0.62) = 0.2676$.

Part (d): State a correct conclusion, using the result of the statistical test, in the context of the problem.

Since the p-value $= 0.2676$ is larger than any reasonable significance level (e.g., $\alpha = 0.05$), we cannot reject the company's claim. That is, we do not have statistically significant evidence to support the student's belief that the proportion of cereal boxes with vouchers is less than 20 percent.

A French study was conducted in the 1990s to compare the effectiveness of using an instrument called a cardiopump with the effectiveness of using traditional cardiopulmonary resuscitation (CPR) in saving lives of heart attack victims. Heart attack patients in participating cities were treated with either a cardiopump or CPR, depending on whether the individual's heart attack occurred on an even-numbered or an odd-numbered day of the month. Before the start of the study, a coin was tossed to determine which treatment, a cardiopump or CPR, was given on the even-numbered days. The other treatment was given on the odd-numbered days. In total, 754 patients were treated with a cardiopump, and 37 survived at least one year; while 746 patients were treated with CPR, and 15 survived at least one year.

(a) The conditions for inference are satisfied in the study. State the conditions and indicate how they are satisfied.

(b) Perform a statistical test to determine whether the survival rate for patients treated with a cardiopump is significantly higher than the survival rate for patients treated with CPR.

Solution

Let A represent the cardiopump treatment, and let B represent the CPR treatment.

Let p_A = proportion of patients who will survive at least one year if treated with the cardiopump.
Let p_B = proportion of patients who will survive at least one year if treated with CPR.

Part (a):

Step 1: State the conditions for inference.

The conditions required for a two-sample z test of equal proportions for an experiment are:
1. Random assignment of treatments to subjects
2. Sufficiently large sample sizes

Step 2: Check the conditions.

1. If we assume that the relevant characteristics of people who have heart attacks on even-numbered and odd-numbered days are comparable, randomly assigning one treatment to be given on even-numbered days and the other to be given on odd-numbered days is a reasonable approximation to randomly assigning the two treatments to the available subjects.
2. The large sample condition is met because all of the following are at least 5 (or 10):

$$n_A\hat{p}_A = 37 \geq 5 \text{ or } 10,\; n_A(1-\hat{p}_A) = 717 \geq 5 \text{ or } 10$$
$$n_B\hat{p}_B = 15 \geq 5 \text{ or } 10,\; n_B(1-\hat{p}_B) = 731 \geq 5 \text{ or } 10$$

Part (b):

Step 1: State a correct pair of hypotheses.

$$H_0: p_A - p_B = 0 \text{ (or } p_A = p_B)$$
$$H_a: p_A - p_B > 0 \text{ (or } p_A > p_B)$$

Step 2: Identify a correct test by name or by formula.

$$z = \frac{\hat{p}_A - \hat{p}_B}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_A} + \frac{1}{n_B}\right)}} = \frac{\hat{p}_A - \hat{p}_B}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n_A} + \frac{\hat{p}(1-\hat{p})}{n_B}}} \text{ where } \hat{p} = \frac{n_A\hat{p}_A + n_B\hat{p}_B}{n_A + n_B}.$$

Step 3: Correct mechanics, including the value of the test statistic and p-value (or rejection region).

$$\hat{p}_A = \frac{37}{754} \approx 0.049 \qquad \hat{p}_B = \frac{15}{746} \approx 0.020 \qquad \hat{p} = \frac{37+15}{754+746} = \frac{52}{1500} \approx 0.035$$

$$z = \frac{\frac{37}{754} - \frac{15}{746}}{\sqrt{\frac{52}{1500}\left(1-\frac{52}{1500}\right)\left(\frac{1}{754}+\frac{1}{746}\right)}} \approx 3.066$$

The p-value is 0.0011.

Step 4: State a correct conclusion in the context of the problem, using the result of the statistical test.

Because the p-value of 0.0011 is very small, that is, less than any reasonable significance level such as $\alpha = 0.01$, or $\alpha = 0.05$, we reject the null hypothesis. We have strong evidence to support the conclusion that the proportion of patients who survive when treated with the cardiopump is higher than the proportion of patients who survive when treated with CPR; that is, the survival rate is higher for patients treated with the cardiopump. (*OR*, If all of these patients had been assigned the cardiopump, we have strong evidence that the survival rate would be higher than if all of these patients had been assigned CPR.)

For many years, the medically accepted practice of giving aid to a person experiencing a heart attack was to have the person who placed the emergency call administer chest compression (CC) plus standard mouth-to-mouth resuscitation (MMR) to the heart attack patient until the emergency response team arrived. However, some researchers believed that CC alone would be a more effective approach.

In the 1990s a study was conducted in Seattle in which 518 cases were randomly assigned to treatments: 278 to CC plus standard MMR and 240 to CC alone. A total of 64 patients survived the heart attack: 29 in the group receiving CC plus standard MMR, and 35 in the group receiving CC alone. A test of significance was conducted on the following hypotheses.

H_0: The survival rates for the two treatments are equal.

H_a: The treatment that uses CC alone produces a higher survival rate.

This test resulted in a *p*-value of 0.0761.

(a) Interpret what this *p*-value measures in the context of this study.

(b) Based on this *p*-value and study design, what conclusion should be drawn in the context of this study? Use a significance level of $\alpha = 0.05$.

(c) Based on your conclusion in part (b), which type of error, Type I or Type II, could have been made? What is one potential consequence of this error?

Solution

Part (a):

The p-value of 0.0761 measures the chance of observing a difference between the two sample proportions $\left(\hat{p}_{CC} - \hat{p}_{CC+MMR}\right)$ as large as or larger than the one observed, if the survival rates for the two treatments (CC alone and CC + MMR) are in fact the same.

Part (b):

Because the p-value of 0.0761 is greater than 0.05, the null hypothesis should not be rejected. That is, there is not sufficient evidence to conclude that the treatment "CC alone" produces a higher survival rate than the standard treatment "CC + MMR."

Part (c):

Because the null hypothesis was not rejected, a Type II error could have occurred. A possible consequence is that CC + MMR would continue as the accepted practice when, in fact, CC alone would result in a higher survival rate.

A large company has two shifts—a day shift and a night shift. Parts produced by the two shifts must meet the same specifications. The manager of the company believes that there is a difference in the proportions of parts produced within specifications by the two shifts. To investigate this belief, random samples of parts that were produced on each of these shifts were selected. For the day shift, 188 of its 200 selected parts met specifications. For the night shift, 180 of its 200 selected parts met specifications.

(a) Use a 96 percent confidence interval to estimate the difference in the proportions of parts produced within specifications by the two shifts.

(b) Based only on this confidence interval, do you think that the difference in the proportions of parts produced within specifications by the two shifts is significantly different from 0 ? Justify your answer.

Solution

Part (a):

Step 1: Identifies the appropriate confidence interval by name or formula and checks appropriate conditions.

Two sample z confidence interval for $p_D - p_N$, the difference in the proportions of parts meeting specifications for the two shifts OR $(\hat{p}_D - \hat{p}_N) \pm z^* \sqrt{\frac{\hat{p}_D(1-\hat{p}_D)}{n_D} + \frac{\hat{p}_N(1-\hat{p}_N)}{n_N}}$.

Conditions:
1. Independent random samples from two separate populations
2. Large samples, so normal approximation can be used

The problem states that random samples of parts were selected from the two different shifts. We need to assume that these parts were produced independently. That is, each employee works the day shift or night shift, but not both, and the machine quality does not vary over time. Since the sample sizes are both 200 and the number of successes (188 and 180) and the number of failures (12 and 20) for each shift are larger than 10, it is reasonable to use the large sample procedures.

Step 2: Correct mechanics

$$\hat{p}_D = \frac{188}{200} = 0.94 \text{ and } \hat{p}_N = \frac{180}{200} = 0.90$$

$$(0.94 - 0.9) \pm 2.0537\sqrt{\frac{0.94 \times 0.06}{200} + \frac{0.9 \times 0.1}{200}}$$

$$0.04 \pm 2.0537 \times 0.0271$$

$$0.04 \pm 0.0556$$

$$(-0.0156, 0.0956)$$

Step 3: Interpretation

Based on these samples, we can be 96 percent confident that the difference in the proportions of parts meeting specifications for the two shifts is between -0.0156 and 0.0956.

Part (b):

Since zero is in the 96 percent confidence interval, zero is a plausible value for the difference $p_D - p_N$, and we do not have evidence to support the manager's belief. In other words, there does not appear to be a statistically significant difference between the proportions of parts meeting specifications for the two shifts at the $\alpha = 0.04$ level.

An environmental science teacher at a high school with a large population of students wanted to estimate the proportion of students at the school who regularly recycle plastic bottles. The teacher selected a random sample of students at the school to survey. Each selected student went into the teacher's office, one at a time, and was asked to respond yes or no to the following question.

Do you regularly recycle plastic bottles?

Based on the responses, a 95 percent confidence interval for the proportion of all students at the school who would respond yes to the question was calculated as $(0.584, 0.816)$.

(a) How many students were in the sample selected by the environmental science teacher?

(b) Given the method used by the environmental science teacher to collect the responses, explain how bias might have been introduced and describe how the bias might affect the point estimate of the proportion of all students at the school who would respond yes to the question.

(c) The statistics teacher at the high school was concerned about the potential bias in the survey. To obtain a potentially less biased estimate of the proportion, the statistics teacher used an alternate method for collecting student responses. A random sample of 300 students was selected, and each student was given the following instructions on how to respond to the question.

- In private, flip a fair coin.
- If heads, you must respond no, regardless of whether you regularly recycle.
- If tails, please truthfully respond yes or no.

(i) What is the expected number of students from the sample of 300 who would be required to respond no because the coin flip resulted in heads?

(ii) The results of the sample showed that 213 of the 300 selected students responded no. Based on the results of the sample, give a point estimate for the <u>proportion</u> of all students at the high school who would respond <u>yes</u> to the question.

Solution

Part (a):

Using the standard formula for a confidence interval for one proportion, the interval (0.584 to 0.816) is found as follows. $\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ where $\hat{p} = \frac{0.584 + 0.816}{2} = 0.7$, the margin of error is $0.816 - 0.7 = 0.116$, and $z^* = 1.96$.

Solving $1.96\sqrt{\frac{0.7(1-0.7)}{n}} = 0.116$ yields $n = \frac{(1.96)^2(0.7)(1-0.7)}{(0.116)^2} \approx 59.95$. The sample size was 60.

Part (b):

Bias might have been introduced because students responded directly to the environmental science teacher. Because the students would know that an environmental science teacher cares about the environment, they might say yes when they actually don't recycle. This would result in a point estimate that is greater than the proportion of all students who would respond yes to the question.

Part (c):

(i) The expected number is $(300)\left(\frac{1}{2}\right) = 150$.

(ii) The point estimate is based on expecting 150 students to be required to say no and 150 students to truthfully answer the question. Of the 213 answers of no, we expect that $213 - 150 = 63$ were from students who truthfully answered the question. That means we expect that the remaining $150 - 63 = 87$ students truthfully answered the question and responded yes. So the point estimate for the proportion of all students at the high school who would respond yes to the question is $\frac{87}{150} = 0.58$.

A polling agency showed the following two statements to a random sample of 1,048 adults in the United States.

> Environment statement: Protection of the environment should be given priority over economic growth.
>
> Economy statement: Economic growth should be given priority over protection of the environment.

The order in which the statements were shown was randomly selected for each person in the sample. After reading the statements, each person was asked to choose the statement that was most consistent with his or her opinion. The results are shown in the table.

	Environment Statement	Economy Statement	No Preference
Percent of sample	58%	37%	5%

(a) Assume the conditions for inference have been met. Construct and interpret a 95 percent confidence interval for the proportion of all adults in the United States who would have chosen the economy statement.

(b) One of the conditions for inference that was met is that the number who chose the economy statement and the number who did not choose the economy statement are both greater than 10. Explain why it is necessary to satisfy that condition.

(c) A suggestion was made to use a two-sample z-interval for a difference between proportions to investigate whether the difference in proportions between adults in the United States who would have chosen the environment statement and adults in the United States who would have chosen the economy statement is statistically significant. Is the two-sample z-interval for a difference between proportions an appropriate procedure to investigate the difference? Justify your answer.

Solution

Part (a):

The appropriate procedure is a one-sample z-interval for a population proportion. The problem stated the conditions for inference have been met, so they do not need to be checked. A 95 percent confidence interval for the population proportion is given as $\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, which is

$$0.37 \pm 1.96\sqrt{\frac{(0.37)(0.63)}{1{,}048}} \approx 0.37 \pm 0.03 = (0.34, 0.40).$$

We are 95 percent confident that the population proportion of all adults in the U.S. who would have chosen the economy statement is between 0.34 and 0.40.

Part (b):

The condition is necessary because the formula for the confidence interval relies on the fact that the binomial distribution can be approximated by a normal distribution which then results in the sampling distribution of $\hat{p}$ being approximately normal. The approximation does not work well unless both $n\hat{p}$ and $n(1-\hat{p})$ are at least 10.

Part (c):

The suggested procedure is not appropriate because one of the requirements for using a two-sample z-interval for a difference between proportions is that the two proportions are based on two independent samples. In the situation described the two proportions come from a single sample and thus are not independent.

During a flu vaccine shortage in the United States, it was believed that 45 percent of vaccine-eligible people received flu vaccine. The results of a survey given to a random sample of 2,350 vaccine-eligible people indicated that 978 of the 2,350 people had received flu vaccine.

(a) Construct a 99 percent confidence interval for the proportion of vaccine-eligible people who had received flu vaccine. Use your confidence interval to comment on the belief that 45 percent of the vaccine-eligible people had received flu vaccine.

(b) Suppose a similar survey will be given to vaccine-eligible people in Canada by Canadian health officials. A 99 percent confidence interval for the proportion of people who will have received flu vaccine is to be constructed. What is the smallest sample size that can be used to <u>guarantee</u> that the margin of error will be less than or equal to 0.02 ?

Solution

Part (a):

Step 1: Identifies the appropriate confidence interval by name or formula and checks appropriate conditions

One-sample (or large-sample) interval for p (the proportion of the vaccine-eligible people in the United States who actually got vaccinated) or $\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

Conditions:
1. Random sample
2. Large sample ($n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$)

The stem of the problem indicates that a random sample of vaccine-eligible people was surveyed. The number of successes (978 vaccine-eligible people who received the vaccine), and failures (1,372 vaccine-eligible people who did not receive the vaccine), are both much larger than 10, so the large-sample interval procedure can be used.

Step 2: Correct mechanics

$$\left(\frac{978}{2{,}350}\right) \pm 2.57583\sqrt{\frac{0.41617(1-0.41617)}{2{,}350}}$$
$$0.41617 \pm 2.57583 \times 0.01017$$
$$0.41617 \pm 0.02619$$
$$(0.38998, 0.44236)$$

Step 3: Interpretation

Based on the sample, we are 99 percent confident that the proportion of the vaccine-eligible people in the United States who actually got vaccinated is between 0.39 and 0.44. Because 0.45 is not in the 99 percent confidence interval, it is not a plausible value for the population proportion of vaccine-eligible people who received the vaccine. In other words, the confidence interval is inconsistent with the belief that 45 percent of those eligible got vaccinated.

Part (b):

The sample-size calculation uses 0.5 as the value of the proportion in order to provide the minimum required sample size to guarantee that the resulting interval will have a margin of error no larger than 0.02.

$$n \geq \frac{(2.576)^2(0.5)(0.5)}{(0.02)^2} = \left(\frac{2.576}{2(0.02)}\right)^2 = 4{,}147.36$$

Thus, a sample of at least 4,148 vaccine-eligible people should be taken in Canada.

Every year, each student in a nationally representative sample is given tests in various subjects. Recently, a random sample of 9,600 twelfth-grade students from the United States were administered a multiple-choice United States history exam. One of the multiple-choice questions is below. (The correct answer is C.)

> In 1935 and 1936 the Supreme Court declared that important parts of the New Deal were unconstitutional. President Roosevelt responded by threatening to
>
> (A) impeach several Supreme Court justices
> (B) eliminate the Supreme Court
> (C) appoint additional Supreme Court justices who shared his views
> (D) override the Supreme Court's decisions by gaining three-fourths majorities in both houses of Congress

Of the 9,600 students, 28 percent answered the multiple-choice question correctly.

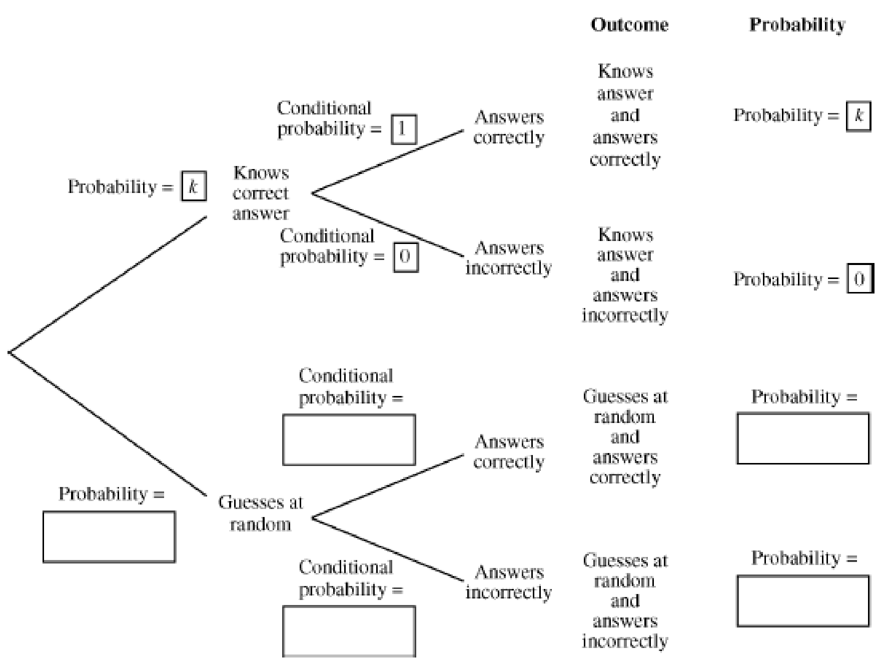
(a) Let p be the proportion of all United States twelfth-grade students who would answer the question correctly. Construct and interpret a 99 percent confidence interval for p.

Assume that students who actually know the correct answer have a 100 percent chance of answering the question correctly, and students who do not know the correct answer to the question guess completely at random from among the four options.

Let k represent the proportion of all United States twelfth-grade students who actually know the correct answer to the question.

(b) A tree diagram of the possible outcomes for a randomly selected twelfth-grade student is provided below. Write the correct probability in each of the five empty boxes. Some of the probabilities may be expressions in terms of k.

TREE DIAGRAM OF OUTCOMES FOR A RANDOMLY SELECTED TWELFTH-GRADE STUDENT



(c) Based on the completed tree diagram, express the probability, in terms of k, that a randomly selected twelfth-grade student would correctly answer the history question.

(d) Using your interval from part (a) and your answer to part (c), calculate and interpret a 99 percent confidence interval for k, the proportion of all United States twelfth-grade students who actually know the answer to the history question. You may assume that the conditions for inference for the confidence interval have been checked and verified.

Solution

Part (a):

The appropriate inference procedure is a one-sample z-interval for a population proportion p, where p is the proportion of all United States twelfth-grade students who would answer the question correctly.

The conditions for this inference procedure are satisfied because:

1. The question states that the students are a random sample from the population, and
2. $n \times \hat{p} = 9{,}600 \times 0.28 = 2{,}688$ and $n \times (1 - \hat{p}) = 9{,}600 \times 0.72 = 6{,}912$ are both much larger than 10.

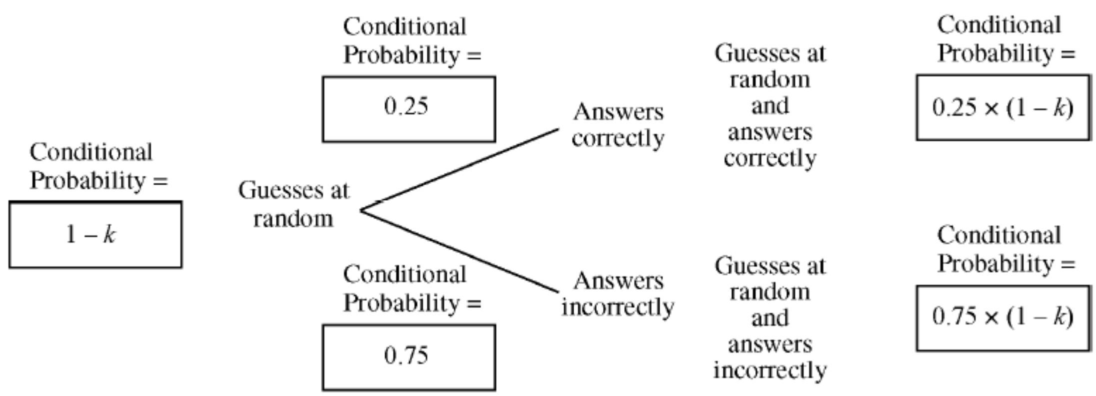
A 99 percent confidence interval for the population proportion p is constructed as follows:

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.28 \pm 2.576\sqrt{\frac{0.28(0.72)}{9{,}600}}$$
$$= 0.28 \pm 0.012$$
$$\rightarrow (0.268, 0.292)$$

We are 99 percent confident that the interval from 0.268 to 0.292 contains the population proportion of all United States twelfth-grade students who would answer this question correctly.

Part (b):

The five probabilities to be filled in the boxes are shown below.



Part (c):

P(answers correctly) =
P(knows correct answer and answers correctly) + P(guesses at random and answers correctly) =
$k + 0.25 \times (1-k)$, which simplifies to $0.25 + 0.75k$, or $\frac{3k+1}{4}$.

Part (d):

We want to estimate k, the proportion of all United States twelfth-grade students who actually know the answer to the history question.

From part (c) the probability that a randomly selected student correctly answers the question is $0.25 + 0.75k$. From part (a) we are 99 percent confident that this probability is between 0.268 and 0.292. Thus the endpoints for a confidence interval for k can be found by equating the expression $0.25 + 0.75k$ from part (c) to the endpoints of the interval from part (a) as follows:

$$0.25 + 0.75k = 0.268 \qquad\qquad 0.25 + 0.75k = 0.292$$
$$k = 0.024 \qquad\qquad k = 0.056$$

We are 99 percent confident that the interval from 0.024 to 0.056 contains the proportion of all United States twelfth-grade students who actually know the answer to the history question.

Sunshine Farms wants to know whether there is a difference in consumer preference for two new juice products—Citrus Fresh and Tropical Taste. In an initial blind taste test, 8 randomly selected consumers were given unmarked samples of the two juices. The product that each consumer tasted first was randomly decided by the flip of a coin. After tasting the two juices, each consumer was asked to choose which juice he or she preferred, and the results were recorded.

(a) Let p represent the population proportion of consumers who prefer Citrus Fresh. In terms of p, state the hypotheses that Sunshine Farms is interested in testing.

(b) One might consider using a one-proportion z-test to test the hypotheses in part (a). Explain why this would not be a reasonable procedure for this sample.

(c) Let X represent the number of consumers in the sample who prefer Citrus Fresh. Assuming there is no difference in consumer preference, find the probability for each possible value of X. Record the x-values and the corresponding probabilities in the table below.

x	$p(x)$

(d) When testing the hypotheses in part (a), Sunshine Farms will conclude that there is a consumer preference if too many or too few individuals prefer Citrus Fresh. Based on your probabilities in part (c), is it possible for the significance level (probability of rejecting the null hypothesis when it is true) for this test to be exactly 0.05 ? Justify your answer.

(e) The preference data for the 8 randomly selected consumers are given in the table below.

Individual	Juice Preference
1	Tropical Taste
2	Citrus Fresh
3	Tropical Taste
4	Tropical Taste
5	Tropical Taste
6	Citrus Fresh
7	Tropical Taste
8	Tropical Taste

Based on these preferences and your previous work, test the hypotheses in part (a).

(f) Sunshine Farms plans to add one of these two new juices—Citrus Fresh or Tropical Taste—to its production schedule. A follow-up study will be conducted to decide which of the two juices to produce. Make one recommendation for the follow-up study that would make it better than the initial study. Provide a statistical justification for your recommendation in the context of the problem.

Solution

Part (a):

$H_0 : p = 0.5$ versus $H_a : p \neq 0.5$

Part (b):

The conditions for the large sample one-proportion z-test are not satisfied. $np = n(1 - p) = 8 \times 0.5 = 4 < 5$.

Part (c):

X will follow a binomial distribution with $n = 8$ and $p = 0.5$. The possible values of X and their corresponding probabilities are given in the table below.

X	Probability
0	0.00391
1	0.03125
2	0.10937
3	0.21875
4	0.27344
5	0.21875
6	0.10937
7	0.03125
8	0.00391

Part (d):

No, there is no possible test with a p-value of exactly 0.05.

The probability that none of the individuals $(X = 0)$ or all of the individuals $(X = 8)$ prefer Citrus Fresh is $2 \times 0.003906 = 0.0078$, which is less than 0.05.

The probability that one or fewer of the individuals $(X \leq 1)$ or seven or more of the individuals $(X \geq 7)$ prefer Citrus Fresh is $2 \times (0.003906 + 0.031250) = 0.070312$, which is greater than 0.05.

Part (e):

For the preference data provided, $X = 2$. From the table of binomial probabilities computed in part (c), the probability that two or fewer of the individuals $(X \leq 2)$ or six or more of the individuals $(X \geq 6)$ prefer Citrus Fresh when $p = 0.5$ is $2 \times (0.003906 + 0.031250 + 0.109375) = 0.289062$. Because the p-value of 0.289062 is greater than any reasonable significance level, say 0.070312, we would not reject the null hypothesis that $p = 0.5$. That is, we do not have statistically significant evidence for a consumer preference between Citrus Fresh and Tropical Taste.

Part (f):

Increase the number of consumers involved in the preference test. More consumers will give you more data, and you will be better able to detect a difference between the population proportion of consumers who prefer Citrus Fresh and 0.5. The sample proportion in the initial study was only 0.25 (2/8), but we were not able to reject the null hypothesis that $p = \frac{1}{2}$. By increasing the number of consumers, a difference of that magnitude would allow the null hypothesis to be rejected. For example, with $n = 80$ and $X = 20$ the large sample z-statistic would be $z = \frac{0.25 - 0.5}{\sqrt{\frac{(0.5)(0.5)}{80}}} = -4.47$ and the p-value would be approximately zero.

In order to monitor the populations of birds of a particular species on two islands, the following procedure was implemented.

Researchers captured an initial sample of 200 birds of the species on Island A; they attached leg bands to each of the birds, and then released the birds. Similarly, a sample of 250 birds of the same species on Island B was captured, banded, and released. Sufficient time was allowed for the birds to return to their normal routine and location.

Subsequent samples of birds of the species of interest were then taken from each island. The number of birds captured and the number of birds with leg bands were recorded. The results are summarized in the following table.

	Island A	Island B
Number Captured in Subsequent Sample	180	220
Number with Leg Bands in Subsequent Sample	12	35

Assume that both the initial sample and the subsequent samples that were taken on each island can be regarded as random samples from the population of birds of this species.

(a) Do the data from the subsequent samples indicate that there is a difference in proportions of the banded birds on these two islands? Give statistical evidence to support your answer.

(b) Researchers can estimate the total number of birds of this species on an island by using information on the number of birds in the initial sample and the proportion of banded birds in the subsequent sample. Use this information to estimate the total number of birds of this species on Island A. Show your work.

(c) The analyses in parts (a) and (b) assume that the samples of birds captured in both the initial and subsequent samples can be regarded as random samples of the population of birds of this species that live on the respective islands. This is a common assumption made by wildlife researchers. Describe two concerns that should be addressed before making this assumption.

Solution

Part (a):

Step 1: States a correct pair of hypotheses

Let p_A = proportion of banded birds on island A
p_B = proportion of banded birds on island B

H_0: $p_A - p_B = 0$ H_a: $p_A - p_B \neq 0$	OR	H_0: $p_A = p_B$ H_a: $p_A \neq p_B$

Step 2: Identifies a correct test (by name or by formula) and checks appropriate conditions.
Two-sample test for proportions

$$z = \frac{\hat{p}_A - \hat{p}_B}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}}}$$

where

$$\hat{p} = \frac{n_A\hat{p}_A + n_B\hat{p}_B}{n_A + n_B} = \frac{12+25}{180+220} = 0.1175$$

Assumptions: independent random samples and large sample sizes

The problem states that it is reasonable to regard the samples as random samples. Since the samples are taken on different islands it may be reasonable to assume that they are independent. Since the expected counts are greater than 10 (or 5), the sample sizes are large enough to accurately use a z-test or a chi-square test.

$n_A\hat{p}_A = 12 \quad n_A(1-\hat{p}_A) = 168$ $n_B\hat{p}_B = 35 \quad n_B(1-\hat{p}_B) = 185$	These estimates of expected counts are used to check the accuracy of normal approximations to two different binomial distributions when the standard error is not estimated from the pooled estimate of the banding probability.
$n_A\hat{p} = 21.15 \quad n_A(1-\hat{p}) = 158.85$ $n_B\hat{p} = 25.85 \quad n_B(1-\hat{p}) = 194.15$	These estimates of expected counts are used to assess the accuracy if a chi-square approximation or a normal approximation to the z-statistic when the standard deviation is estimated using the pooled estimate of the banding probability.

Step 3: Correct mechanics, including the value of the test statistic and P-value (or rejection region)

$$\hat{p}_A = \frac{12}{180} = 0.067 \quad \text{and} \quad \hat{p}_B = \frac{35}{220} = 0.159$$

$$\hat{p} = \frac{12+35}{180+220} = \frac{47}{400} = .1175$$

$$z = \frac{0.067 - 0.159}{\sqrt{\frac{(0.1175)(0.8825)}{180} + \frac{(0.1175)(0.8825)}{220}}} = \frac{-0.092}{\sqrt{0.00105}} = \frac{-0.092}{0.032} = -2.875$$

p-value = 0.00429

Step 4: Stating a correct conclusion in the context of the problem, using the result of the statistical test.

Since p-value=0.00429 is smaller than $\alpha = 0.05$, reject the null hypothesis. There is convincing evidence that the proportions of banded birds on the two islands are not the same.

Part (b):

For island A,

$n_I = 200$ where n_I is the number of birds banded in the initial sample,

$\hat{p}_S = \frac{12}{180} = .067$ where $\hat{p}_S$ is the proportion banded in the subsequent sample.

We expect that the proportion of banded birds in the subsequent sample is approximately equal to the proportion of the population that is banded. Then

$$\hat{p}_S \approx \frac{n_I}{\text{population size}}$$

and the population size can be estimated by

$$\text{estimated population size} = \frac{n_I}{\hat{p}_S} = \frac{200}{0.06667} \approx 3000$$

The Blue Shell Shuttle Bus Company has recently acquired the rights to run a shuttle between Lonestar's hotels and its airport, which is several miles away. For the new route, the company has a choice of running coaches that can carry up to 60 people or smaller vans that can carry up to 12 people. The company has a policy that each of its routes is served only by one type of shuttle vehicle. In addition, due to the allocation of their vehicles to other routes, no change in their decision can be considered for at least a year. The annual return (profit or loss) depends on whether the demand for the service is strong or weak. Research suggests that the following returns can be expected.

Annual Return ($10,000)		
Vehicle Decision	Demand	
	Strong	Weak
Coach	84	–27
Van	61	45

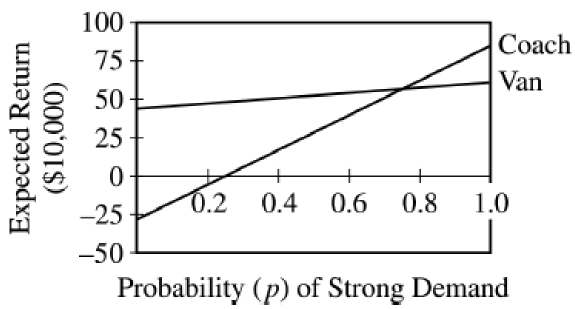
For instance, if a coach is used and demand is strong, the expected annual return is $840,000. The expected return to the company can be calculated based on the probability of a strong demand. Let p represent the probability of strong demand; then $(1 - p)$ represents the probability of weak demand.

An equation that can be used to compute the expected return from the use of coaches based on the value of p is

$$84p + (-27)(1 - p) = 111p - 27.$$

An equation that can be used to compute the expected return from the use of vans based on the value of p is

$$61p + 45(1 - p) = 16p + 45.$$



(a) The value of p for which the expected annual return for the vans is equal to the expected annual return for the coaches is 0.76. If the probability of strong demand is less than this value, which decision, running coaches or running vans, will provide the greater expected return? Justify your answer.

(b) There are several thousand markets similar to Lonestar's market across the country. A random sample of 100 of these markets reveals that the demand for an airport shuttle is strong in 65 of them and the demand in the remaining 35 is weak. Using the results of this sample, construct and interpret a 95 percent confidence interval for the proportion of similar markets that will experience a strong demand.

(c) The president of Blue Shell has decided to use vans for the new route. Using the results of the analysis in parts (a) and (b), write a few sentences to justify this decision.

(d) After looking at the interval in part (b) and considering possible annual returns, the vice president of Blue Shell believes that the president has made an incorrect decision in choosing to use vans. Explain how this conflicting position could be supported.

Part (a):

If the probability of a strong market is less than 0.76, vans would provide a greater expected annual return because

Graphical Argument
the expected annual return for vans is higher, as can be seen in the graph — the line for the expected return for vans is above the line for the expected return for coaches for all values of p less than 0.76

OR

Algebraic Argument
the expected return for coaches and the expected return for vans are both linear functions of the probability of strong demand (p). These functions are equal at $p = 0.76$. For any probability less than 0.76, vans have a higher expected return than coaches. For example, since the expected return for vans when $p = 0.5$ (\$530,000) is greater than the expected return for coaches when $p = 0.5$ (\$285,000), the expected return for vans must be higher when $p < 0.76$.

Part (b):

Step 1: Identifies appropriate confidence interval by name or by formula.

95% confidence interval for p = proportion of similar markets that will experience strong demand.

OR

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Step 2: States and checks appropriate conditions.

Conditions: random sample, sample size < 10% of the population size, and large sample size.

Check for large sample size:
$\hat{p} = 0.65 \qquad n\hat{p} = 65 \qquad n(1-\hat{p}) = 35$

Both $n\hat{p}$ and $n(1-\hat{p})$ are greater than 10 (or 5), so the sample size is large enough to proceed.

Step 3: Computes mechanics correctly.

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.65 \pm 1.96\sqrt{\frac{(0.65)(0.35)}{100}} = (0.55652, 0.74348)$$

Step 4: Interprets the confidence interval <u>in context</u>.

We can be 95% confident that the true proportion of similar markets that experience strong demand is between 0.56 and 0.74.

Part (c):

Based on the interval in part (b), the plausible values for the probability of strong demand are between 0.56 and 0.74. As discussed in part (a), the expected annual return is greater for vans than for coaches for all values in this range. Therefore, vans should be used for the new route.

Part (d):

The demand in Lonestar's market will either be strong or weak. We think that the demand is more likely to be strong than weak because the entire interval in part (b) is above 0.5. As shown in the table, if demand is strong, coaches will produce an annual return of \$840,000, while vans will only produce an annual return of \$610,000. Since the demand is more likely to be strong than weak and the annual return from coaches is much higher than that of vans in a strong market, coaches would be the best choice for the new route.

Each person in a random sample of 1,026 adults in the United States was asked the following question.

"Based on what you know about the Social Security system today, what would you like Congress and the President to do during this next year?"

The response choices and the percentages selecting them are shown below.

Completely overhaul the system	19%
Make some major changes	39%
Make some minor adjustments	30%
Leave the system the way it is now	11%
No opinion	1%

(a) Find a 95% confidence interval for the proportion of all United States adults who would respond "Make some major changes" to the question. Give an interpretation of the confidence interval and give an interpretation of the confidence level.

(b) An advocate for leaving the system as it is now commented, "Based on this poll, only 39% of adults in the sample responded that they want some major changes made to the system, while 41% responded that they want only minor changes or no changes at all. Therefore, we should not change the system." Explain why this statement, while technically correct, is misleading.

Solution

Part (a):

Large sample confidence interval for a population proportion.

Requires large population (N $\geq$ 20n) and large sample. Can consider sample large since

$$n\hat{p} = 1026(.39) = 400.14 \geq 10, \quad n(1-\hat{p}) = 1026(.61) = 625.86 \geq 10$$

OR $\quad n\,\hat{p} \geq 5,\ n(1-\hat{p}) \geq 5$

OR $\quad \hat{p} \pm 3\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = .39 \pm .0298 = (.3602, .4198)$ is within the interval (0, 1).

Computation of 95 percent confidence interval for one population proportion:

$$.39 \pm 1.96\sqrt{\frac{(.39)(.61)}{1026}} = .39 \pm .0298 = (.3602, .4198)$$

If use the calculator with x = 400, get (.36002, .41971)

Interpretation of confidence interval:

Based on the sample, we estimate that the proportion of U.S. adults who would respond "make some changes" is between .36 and .42.

OR

We are 95 percent confident that the proportion of all U.S. adults who would respond "make some changes" is between .36 and .42.

Interpretation of the confidence level:

In the long run, 95 percent of the (different) intervals generated using this method will contain the true population proportion.

OR

The *method* used to produce this estimate has a 5 percent error rate.

Part (b):

The statement is misleading because in addition to the 39 percent who wanted major change, 19 percent wanted a complete overhaul of the system. These two groups combined represent 58 percent (more than half) of the sample.

OR

Since the confidence interval from part (a) for the percentage of all U.S. adults who would respond "make some major changes" includes 41 percent, we can't conclude that the two population percentages are different.

OR

The advocate is claiming that we should not change the system on the basis of adding the 30 percent who want to make some minor adjustments to the 11 percent who want to leave the system the way it is now. It is misleading to claim that these 30 percent don't want to change the system.

2. The manager of a local fast-food restaurant is concerned about customers who ask for a water cup when placing an order but fill the cup with a soft drink from the beverage fountain instead of filling the cup with water. The manager selected a random sample of 80 customers who asked for a water cup when placing an order and found that 23 of those customers filled the cup with a soft drink from the beverage fountain.

a) Construct and interpret a 95 percent confidence interval for the proportion of all customers who, having asked for a water cup when placing an order, will fill the cup with a soft drink from the beverage fountain.

b) The manager estimates that each customer who asks for a water cup but fills it with a soft drink costs the restaurant \$0.25. Suppose that in the month of June 3,000 customers ask for a water cup when placing an order. Use the confidence interval constructed in part (a) to give an interval estimate for the cost to the restaurant for the month of June from the customers who ask for a water cup but fill the cup with a soft drink.

Solution

Part (a):

Step 1: Identify the appropriate confidence interval by name or formula and check appropriate conditions.

The appropriate procedure is a one-sample z-interval for a population proportion p. In this case, the population is all customers of the restaurant who ask for a water cup, and p is the proportion of that population who will fill the cup with a soft drink. The appropriate formula is $\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

Conditions:

1. Random sample
2. Large sample (number of successes $n\hat{p} \geq 10$ and number of failures $n(1-\hat{p}) \geq 10$)

For condition 1 the stem of the problem states that a random sample of customers who asked for a water cup was used.

For condition 2 the number of successes (filled cup with soft drink) is 23 and the number of failures is 57, both of which are greater than 10.

Step 2: Correct mechanics

The sample proportion is $\hat{p} = \frac{23}{80} = 0.2875$. The confidence interval is

$$0.2875 \pm 1.96\sqrt{\frac{0.2875(1-0.2875)}{80}}$$
$$= 0.2875 \pm 1.96(0.0506) \qquad \text{or } 0.1883 \text{ to } 0.3867.$$
$$= 0.2875 \pm 0.0992$$

Step 3: Interpretation

We can be 95 percent confident that in the population of all customers of the restaurant who ask for a water cup, the proportion who will fill it with a soft drink is between 0.1883 and 0.3867.

Part (b):

Using the confidence interval in part (a), a 95 percent interval estimate for the number of customers in June who asked for a water cup but then filled it with a soft drink is $3{,}000 \times 0.1883$ to $3{,}000 \times 0.3857$, or 565 to 1,160. At a cost of $0.25 per customer, a 95 percent interval estimate for the cost to the restaurant in June is $141.25 to $290.00.

4. Tumbleweed, commonly found in the western United States, is the dried structure of certain plants that are blown by the wind. Kochia, a type of plant that turns into tumbleweed at the end of the summer, is a problem for farmers because it takes nutrients away from soil that would otherwise go to more beneficial plants. Scientists are concerned that kochia plants are becoming resistant to the most commonly used herbicide, glyphosate. In 2014, 19.7 percent of 61 randomly selected kochia plants were resistant to glyphosate. In 2017, 38.5 percent of 52 randomly selected kochia plants were resistant to glyphosate. Do the data provide convincing statistical evidence, at the level of $\alpha = 0.05$, that there has been an increase in the proportion of all kochia plants that are resistant to glyphosate?

$H_0: P_{2017} - P_{2014} = 0$ $\quad$ P_{2017} = prop. plants resistant in 2017

$H_a: P_{2017} - P_{2014} > 0$ $\quad$ P_{2014} = prop. plants resistant in 2014.

Assumptions/Conditions
- Both samples were random
- samples < 10% of all plants
- For both samples, np and nq are more than 10.

2017: $np = (52)(0.385) = 20$
$nq = (52)(1-0.385) = 32$
2014: $np = (61)(0.197) = 12$
$nq = (61)(1-0.197) = 49$

Because conditions are met, we will conduct a two proportion z hypothesis test.

z test statistic = 2.210
p-value = 0.0136

Note: As long as we state what test we are conducting, in this case a 2-prop z-test, we can use technology to get the test statistic and p-value.

Conclusion: Because the p-value is less than $\alpha = 0.05$, we reject the null hypothesis. We have evidence that there has been an increase in proportion of Kochia plants from 2014 to 2017 that are resistant to glyphosphate.