AP FRQ Review – Mr. Rich **Name:______________________________** **Per:_____** **Seat:____**

AP Calculus AB

Topic 7: Implicit Differentiation

Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

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(a) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(-1,\,1)} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{4}$

An equation for the tangent line is $y = \frac{1}{4}(x+1) + 1$.

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{equation for tangent line} \end{cases}$

(b) $3y^2 - x = 0 \Rightarrow x = 3y^2$

So, $y^3 - xy = 2 \Rightarrow y^3 - (3y^2)(y) = 2 \Rightarrow y = -1$

$(-1)^3 - x(-1) = 2 \Rightarrow x = 3$

The tangent line to the curve is vertical at the point $(3, -1)$.

$3 : \begin{cases} 1 : \text{sets } 3y^2 - x = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{coordinates} \end{cases}$

(c) $\frac{d^2y}{dx^2} = \frac{(3y^2 - x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$

$\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(-1,\,1)} = \frac{(3 \cdot 1^2 - (-1)) \cdot \frac{1}{4} - 1 \cdot \left(6 \cdot 1 \cdot \frac{1}{4} - 1\right)}{(3 \cdot 1^2 - (-1))^2}$

$= \frac{1 - \frac{1}{2}}{16} = \frac{1}{32}$

$4 : \begin{cases} 2 : \text{implicit differentiation} \\ 1 : \text{substitution for } \frac{dy}{dx} \\ 1 : \text{answer} \end{cases}$

Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{4x - 2xy}{x^2 + y^2 + 1}$.

(b) Write an equation of each horizontal tangent line to the curve.

(c) The line through the origin with slope -1 is tangent to the curve at point P. Find the x– and y–coordinates of point P.

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(a) $6y^2\frac{dy}{dx} + 6x^2\frac{dy}{dx} + 12xy - 24x + 6\frac{dy}{dx} = 0$

$\frac{dy}{dx}(6y^2 + 6x^2 + 6) = 24x - 12xy$

$\frac{dy}{dx} = \frac{24x - 12xy}{6x^2 + 6y^2 + 6} = \frac{4x - 2xy}{x^2 + y^2 + 1}$

2 { 1: implicit differentiation
1: verifies expression for $\frac{dy}{dx}$ }

(b) $\frac{dy}{dx} = 0$

$4x - 2xy = 2x(2 - y) = 0$

$x = 0$ or $y = 2$

When $x = 0$, $2y^3 + 6y = 1$; $y = 0.165$

There is no point on the curve with y coordinate of 2.

$y = 0.165$ is the equation of the only horizontal tangent line.

4 { 1: sets $\frac{dy}{dx} = 0$
1: solves $\frac{dy}{dx} = 0$
1: uses solutions for x to find equations of horizontal tangent lines
1: verifies which solutions for y yield equations of horizontal tangent lines }

Note: max 1/4 [1-0-0-0] if $dy/dx = 0$ is not of the form $g(x, y)/h(x, y) = 0$ with solutions for both x and y

(c) $y = -x$ is equation of the line.

$2(-x)^3 + 6x^2(-x) - 12x^2 + 6(-x) = 1$

$-8x^3 - 12x^2 - 6x - 1 = 0$

$x = -1/2, \quad y = 1/2$

–or–

$\frac{dy}{dx} = -1$

$4x - 2xy = -x^2 - y^2 - 1$

$4x + 2x^2 = -x^2 - x^2 - 1$

$4x^2 + 4x + 1 = 0$

$x = -1/2, \quad y = 1/2$

3 { 1: $y = -x$
1: substitutes $y = -x$ into equation of curve
1: solves for x and y }

–or–

3 { 1: sets $\frac{dy}{dx} = -1$
1: substitutes $y = -x$ into $\frac{dy}{dx}$
1: solves for x and y }

Note: max 2/3 [1-1-0] if importing incorrect derivative from part (a)

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

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(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\frac{dy}{dx} - 3x^2y - x^3\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$

$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$

$2\begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verifies expression for } \frac{dy}{dx} \end{cases}$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\frac{dy}{dx} = \frac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\frac{dy}{dx} = \frac{-6-4}{-4-1} = \frac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x-1)$

$4\begin{cases} 1: \ y^2 - y = 6 \\ 1: \ \text{solves for } y \\ 2: \ \text{tangent lines} \end{cases}$

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x\left(2y - x^2\right) = 0$ gives $x = 0$ or $y = \frac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \frac{1}{2}x^2$, $\frac{1}{4}x^5 - \frac{1}{2}x^5 = 6$

$-\frac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

$3\begin{cases} 1: \ \text{sets denominator of } \frac{dy}{dx} \text{ equal to 0} \\ 1: \ \text{substitutes } y = \frac{1}{2}x^2 \text{ or } x = \pm\sqrt{2y} \\ \quad\ \text{into the equation for the curve} \\ 1: \ \text{solves for } x\text{-coordinate} \end{cases}$

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

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(a) $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6 - 2x) - 2y^2$

$= 2y^3(6 - 2x)^2 - 2y^2$

$\left.\frac{d^2y}{dx^2}\right|_{\left(3, \frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$

3 :
- 2 : $\frac{d^2y}{dx^2}$
  - < −2 > product rule or chain rule error
- 1 : value at $\left(3, \frac{1}{4}\right)$

(b) $\frac{1}{y^2}\,dy = (6 - 2x)\,dx$

$-\frac{1}{y} = 6x - x^2 + C$

$-4 = 18 - 9 + C = 9 + C$

$C = -13$

$y = \frac{1}{x^2 - 6x + 13}$

6 :
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $f(3) = \frac{1}{4}$
- 1 : solves for y

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

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Solution	Scoring
(a) $2x + 8yy' = 3y + 3xy'$ $(8y - 3x)y' = 3y - 2x$ $y' = \dfrac{3y - 2x}{8y - 3x}$	$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$
(b) $\dfrac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$ When $x = 3$, $3y = 6$ $y = 2$ $3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$ Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.	$3 : \begin{cases} 1 : \dfrac{dy}{dx} = 0 \\ 1 : \text{shows slope is 0 at } (3, 2) \\ 1 : \text{shows } (3, 2) \text{ lies on curve} \end{cases}$
(c) $\dfrac{d^2y}{dx^2} = \dfrac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$ At $P = (3, 2)$, $\dfrac{d^2y}{dx^2} = \dfrac{(16 - 9)(-2)}{(16 - 9)^2} = -\dfrac{2}{7}$. Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.	$4 : \begin{cases} 2 : \dfrac{d^2y}{dx^2} \\ 1 : \text{value of } \dfrac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

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(a) $2yy' = y + xy'$ $(2y - x)y' = y$ $y' = \frac{y}{2y - x}$	$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$
(b) $\frac{y}{2y - x} = \frac{1}{2}$ $2y = 2y - x$ $x = 0$ $y = \pm\sqrt{2}$ $(0, \sqrt{2}), (0, -\sqrt{2})$	$2 : \begin{cases} 1 : \frac{y}{2y - x} = \frac{1}{2} \\ 1 : \text{answer} \end{cases}$
(c) $\frac{y}{2y - x} = 0$ $y = 0$ The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x.	$2 : \begin{cases} 1 : y = 0 \\ 1 : \text{explanation} \end{cases}$